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LETTER TO THE EDITOR

On the remarkable correspondence between generalized Thirring model and linear Klein–Gordon equation

R Z Zhdanov

Institute of Mathematics of the Academy of Science of Ukraine, Tereshchenkivska Street 3, Kiev – 4, Ukraine

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Abstract. We have suggested a substitution, reducing generalized Thirring equations to a linear Klein–Gordon equation. Using the above substitution one can obtain infinitely many exact solutions of the nonlinear Thirring system without applying the inverse scattering method.

When discussing the generalized Thirring model, we mean the following system of nonlinear partial differential equations:

$$\begin{aligned} iu_y &= mv + \lambda_1 |v|^2 u \\ iv_x &= mu + \lambda_2 |u|^2 v \end{aligned} \tag{1}$$

where $u = u(x, y)$, $v = v(x, y)$ are smooth complex-valued functions, m, λ_1, λ_2 are real constants, $|u|^2 = uu^*$ and $|v|^2 = vv^*$.

Provided $\lambda_1 = \lambda_2 = \lambda$ equations (1) coincide with the classical Thirring model that is integrable by means of the inverse scattering method [3]. Not long ago it had been established by David [1] that the generalized Thirring model (1) is also integrable by the said method and, therefore, had soliton solutions.

In this letter we restrict ourselves to the case $\lambda_1 = -\lambda_2 = \lambda$, i.e. we consider the following system:

$$\begin{aligned} iu_y &= mv + \lambda |v|^2 u \\ iv_x &= mu - \lambda |u|^2 v. \end{aligned} \tag{2}$$

To establish correspondence between (2) and the linear Klein–Gordon equation

$$w_{xy} + m^2 w = 0 \tag{3}$$

we apply the ansatz [2]

$$\begin{aligned} u &= F_1 \exp\{iG + iC\} \\ v &= F_2 \exp\{iG - iC\} \end{aligned} \tag{4}$$

where F_1, F_2, G are some real-valued functions, $C \in \mathbb{R}^1$.

Substitution of (4) into (2) yields a system of four nonlinear partial differential equations for F_1, F_2, G:

(i) $F_{1y} = -mF_2 \sin 2C$,

$$\begin{aligned}&\text{(ii)}\ F_{2x}=mF_1\sin 2C\\&\text{(iii)}\ G_x=-mF_1F_2^{-1}\cos 2C+\lambda F_1^2\\&\text{(iv)}\ G_y=-mF_2F_1^{-1}\cos 2C-\lambda F_2^2.\end{aligned}\tag{5}$$

Analysis of the over-determined system (5) shows that in the cases $C\neq\pi n$, $n\in\mathbb{Z}$ and $C\neq\pi/4+\pi n$, $n\in\mathbb{Z}$ its general solution is of the form $F_i=F_i(\alpha x+\beta y)$, $G=G(\alpha x+\beta y)$. Such plane-wave solutions are well known, that is why we restrict ourselves to the cases $C=\pi n$ and $C=\pi/4+\pi n$ (because of periodicity of the functions $\sin 2C$, $\cos 2C$ one can choose $C=0$ and $C=\pi/4$).

The case 1. $C=\pi/4$. In this case system (5) reads:

$$\begin{aligned}&\text{(i)}\ F_{1y}=-mF_2\\&\text{(ii)}\ F_{2x}=mF_1\\&\text{(iii)}\ G_x=\lambda F_1^2\\&\text{(iv)}\ G_y=-\lambda F_2^2.\end{aligned}\tag{6}$$

Since

$$(G_x)_y=2\lambda F_1F_{1y}=-2\lambda mF_1F_2=-2\lambda F_2F_{2x}=(G_y)_x$$

system (6) is compatible and its general solution can be represented in the form

$$F_1=w(x,y)\qquad F_2=-m^{-1}w_y(x,y)$$
$$G=\lambda\int_a^x w^2(\tau,y)\,d\tau-\lambda m^{-2}\int_b^y w_y^2(a,\tau)\,d\tau$$

where a, b are some real constants and $w=w(x,y)$ is an arbitrary solution of (3).

Thus, each solution of the linear equation (3) gives rise to the exact solution of the nonlinear system (2) of the form

$$\begin{aligned}u&=w\exp\left\{\frac{i\pi}{4}+i\lambda\int_a^x w^2(\tau,y)\,d\tau-i\lambda m^{-2}\int_b^y w_y^2(a,\tau)\,d\tau\right\}\\ v&=-m^{-1}w_y\exp\left\{-\frac{i\pi}{4}+i\lambda\int_a^x w^2(\tau,y)\,d\tau-i\lambda m^{-2}\int_b^y w_y^2(a,\tau)\,d\tau\right\}\end{aligned}\tag{7}$$

where $w=w(x,y)$ is an arbitrary solution of (3).

The case 2. $C=0$. From the first pair of equations of system (5) it follows that $F_1=F_1(x)$, $F_2=F_2(y)$. Substituting these expressions into the remaining equations of system (5) and integrating, we get

$$F_1=ax^{-1/2}\qquad F_2=ay^{-1/2}$$
$$G=2m(xy)^{1/2}+\lambda a^2\ln(xy^{-1})\qquad a\in\mathbb{R}^1.$$

The above formulae after being substituted into ansatz (4) with $C=0$ yields the following class of exact solutions of the initial system (2):

$$u=ax^{-1/2}\exp\{-2im(xy)^{1/2}+i\lambda a^2\ln(xy^{-1})\}$$
$$v=ay^{-1/2}\exp\{-2im(xy)^{1/2}+i\lambda a^2\ln(xy^{-1})\}.$$

In conclusion, let us note that by force of invariance of system (2) under the one-parameter group of gauge transformations

$$u' = u \exp(i\theta) \qquad v' = v \exp(i\theta) \qquad \theta \in \mathbb{R}^1$$

solution (7) can be rewritten in the equivalent form

$$u = w \exp\left\{ i\lambda \int_a^x w^2(\tau, y)\, \mathrm{d}\tau = i\lambda m^{-2} \int_b^y w_y^2(a, \tau)\, \mathrm{d}\tau \right\}$$

$$v = im^{-1} w_y \exp\left\{ i\lambda \int_a^x w^2(\tau, y)\, \mathrm{d}\tau - i\lambda m^{-2} \int_b^y w_y^2(a, \tau)\, \mathrm{d}\tau \right\}.$$

References

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